On Decompositions of KdV 2-Solitons

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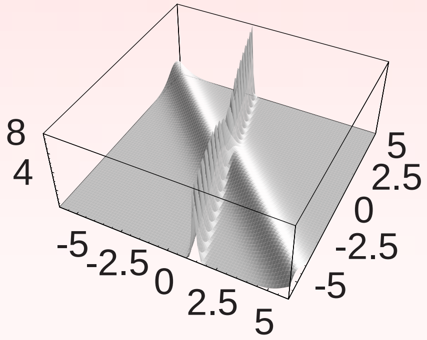


Abstract: There is no deep mathematics here, but a student project collected and collated difficult to find information on this topic. Moreover, we discovered a few new twists. All together, this can help us interpret the "interaction" of KdV solitons.

The KdV Equation

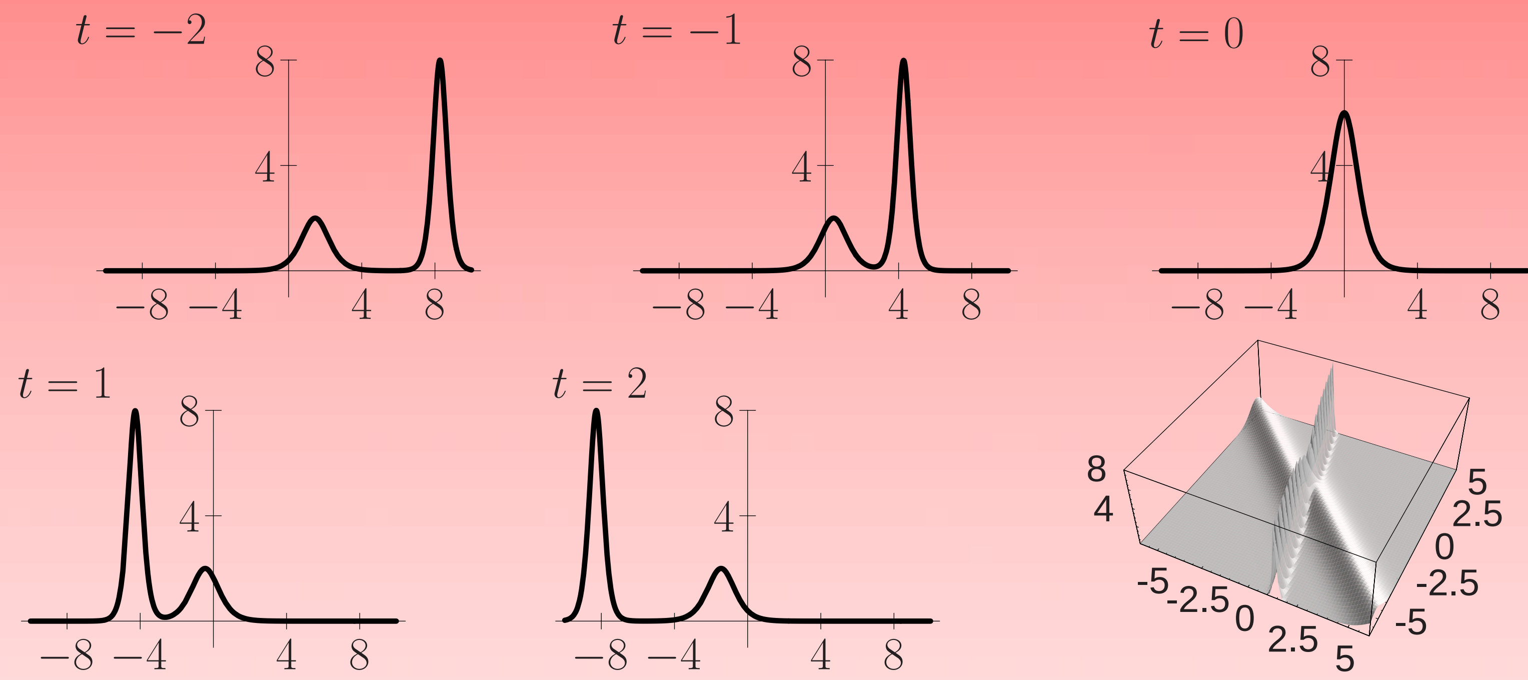
$$u_t - \frac{3}{2}uu_x - \frac{1}{4}u_{xxx} = 0$$

- Originally derived over 100 years ago to model surface waves in a canal.
- Category in the Mathematics Classification Scheme (MCS2000) called “KdV-like equations” (35Q53) and frequently paired with the adjective “ubiquitous”
- Completely Integrable: we can write exact solutions.
- It has “hump-like” travelling wave solution:

$$u_1(x,t) = u_1(x,t;k,\xi) = 2k^2\mathrm{sech}^2(\eta(x,t;k,\xi))$$
$$\eta(x,t;k,\xi) = kx + k^3t + \xi$$

- There are also n-soliton solutions showing nonlinear superposition of a collection of these “humps”:

KdV 2-Soliton



$$u_2(x,t) = 2\partial_x^2 \log(\tau) \qquad \tau = e^{-\eta_1-\eta_2} + e^{\eta_1-\eta_2} + e^{\eta_2-\eta_1} + \epsilon^2 e^{\eta_1+\eta_2}$$

$$\epsilon = \frac{k_2 - k_1}{k_1 + k_2} \qquad \eta_i = \eta(x,t;k_i,\xi_i) = k_i x + k_i^3 t + \xi_i$$

Looks *similar* to a sum of two travelling waves, but it is not! Note:

- Height at $t=0$ not sum of heights. Trajectories are “bent” at time of collision.

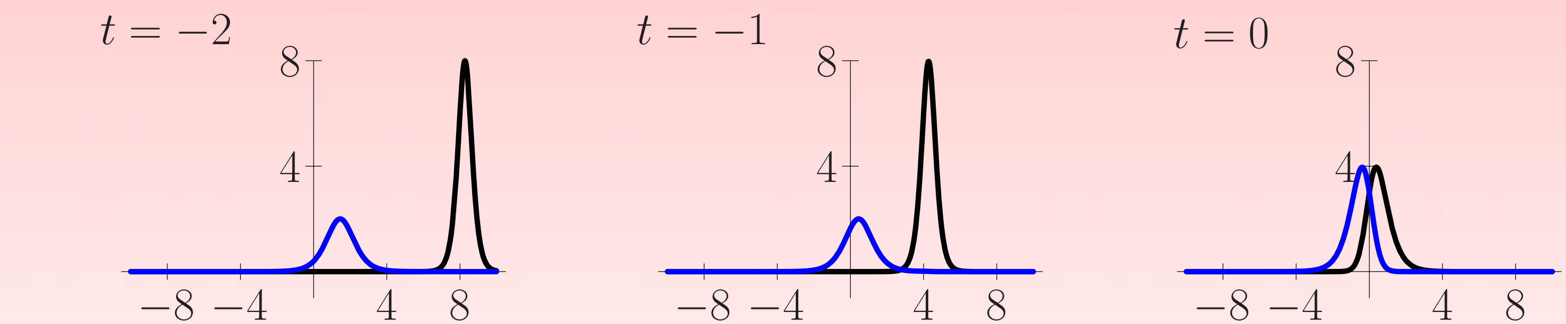
> ***Philosophical Question:*** Does the tall one pass through the small one, or does the trailing one pass its momentum to the first?

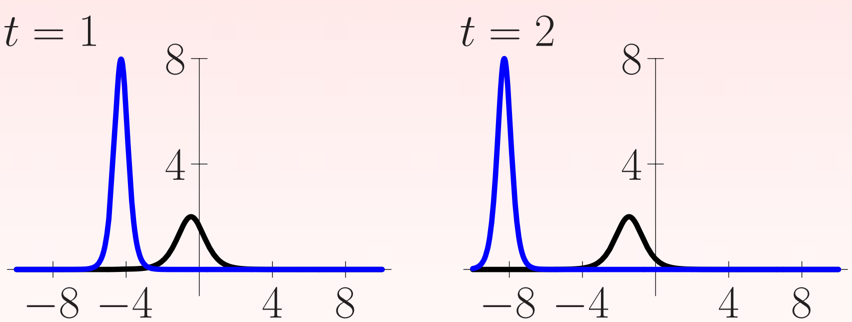
A Decomposition (BKY 2006): $u_2 = f_1 + f_2$

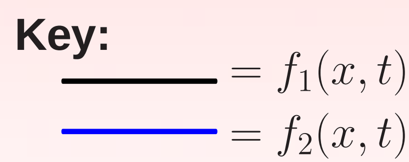
Consider f_1 and f_2 such that $u_2(x,t) = f_1(x,t) + f_2(x,t)$. Clearly, there are many ways to do this, but some are more interesting than others. The following is original to us

$$f_1(x,t) = \frac{8\epsilon^2((k_2+k_1)^2 + k_2^2 e^{2\eta_1} + k_1^2 e^{2\eta_2})}{\tau^2}$$

$$f_2(x,t) = \frac{8((k_2-k_1)^2 + k_2^2 e^{-2\eta_1} + k_1^2 e^{-2\eta_2})}{\tau^2}.$$







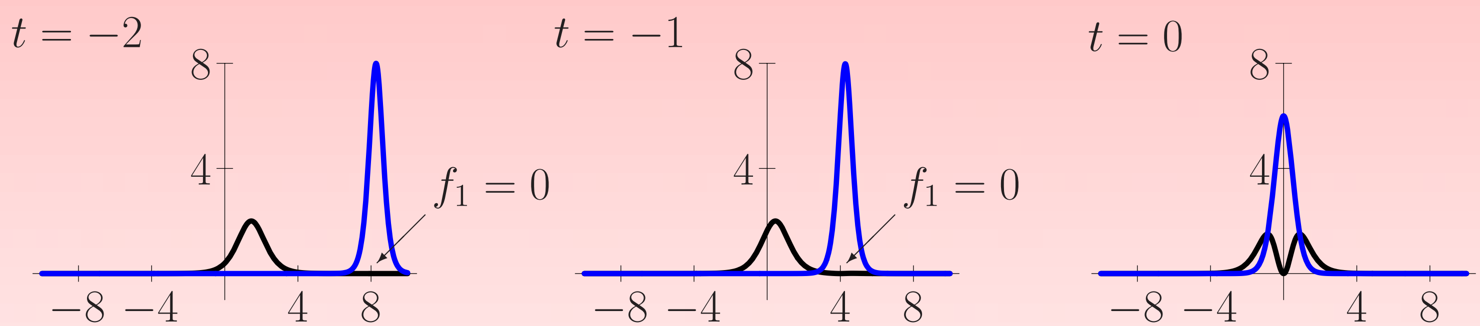
Properties: order preserving, positive formula very nice

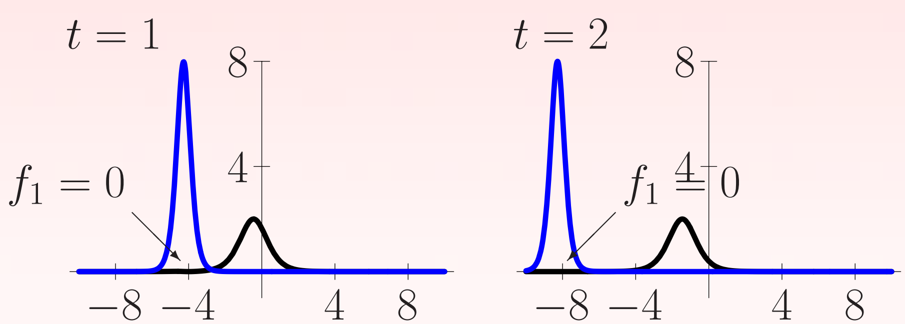
Yoneyama's Speed Preserving Decomposition (1984)

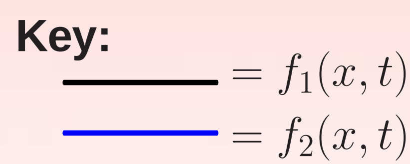
$$f_1 = 2k_1(g(\eta_1,\eta_2))_x \operatorname{sech}^2[g(\eta_1,\eta_2)] \qquad f_2 = 2k_2(g(\eta_2,\eta_1))_x \operatorname{sech}^2[g(\eta_2,\eta_1)]$$

$$g(\eta_i,\eta_j) = \eta_i + \frac{1}{2}\ln\left(\frac{1+\epsilon^2\exp(2\eta_j)}{1+\exp(2\eta_j)}\right).$$

Oldest published decomposition, argued that solitons are *attractive*. Note that f_1 has a *zero* near peak of f_2.







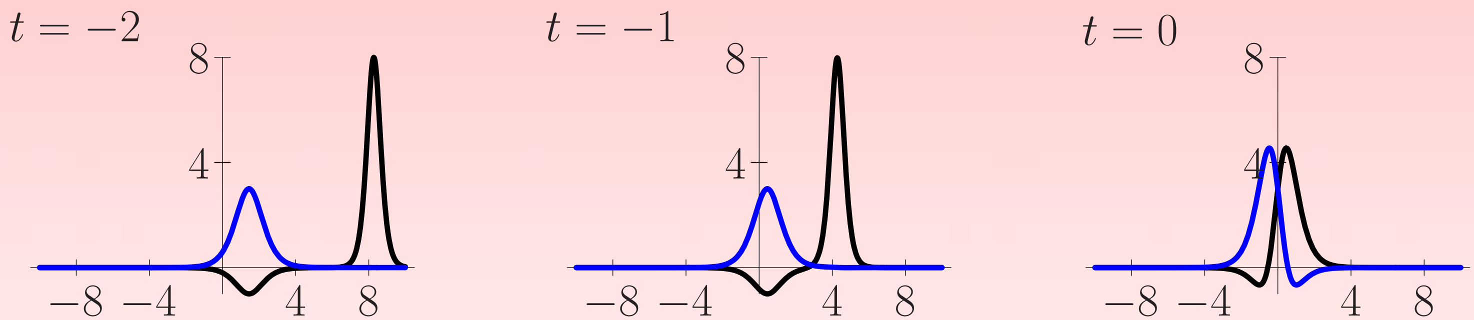
Properties: speed preserving, non-negative ($f_1 = 0$)
formula pretty nice
further developed by Moloney-Hodnett, Campbell-Parks, Fuch

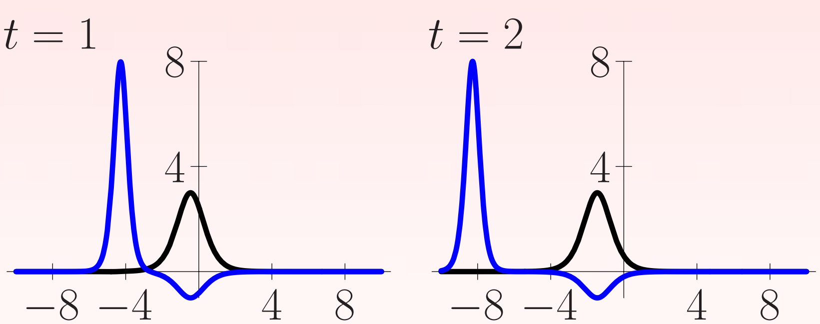
Miller-Christiansen: Order and Mass Preserving

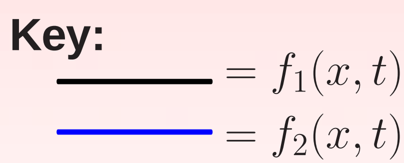
Inspired by Bowtell-Stuart's singularity analysis, present decomposition satisfying:

$$(f_i)_t - \frac{3}{4}\left(u_2(f_i)_x + (u_2)_x f_i\right) - \frac{1}{4}(f_i)_{xxx} = 0.$$

$$f_1 = 4\epsilon^2/\tau^2 \left(k_1(k_1+k_2)^2 k_1 - k_2 e^{-2\eta_2} + 2(k_1+k_2)^2 + 2k_2^2 e^{2\eta_1} + k_1(k_1+k_2)e^{2\eta_2}\right)$$
$$f_2 = 4/\tau^2 \left(k_1(k_1+k_2)e^{-2\eta_2} + 2k_2^2 e^{-2\eta_1} + 2(k_1-k_2)^2 + \epsilon^2 k_1(k_1-k_2)e^{2\eta_2}\right).$$







Properties: order and mass preserving, $\pm$,
components satisfy coupled PDEs
formulas given here for first time!

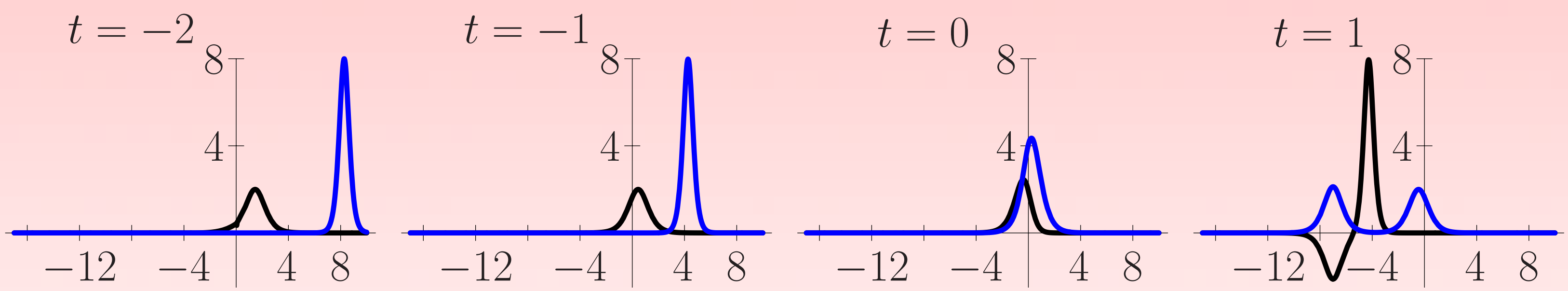
Nguyen's "Ghost" Solitons

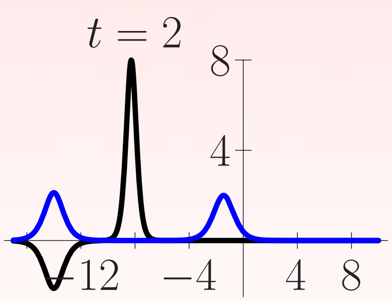
"Ghosts" created at collision travel *ahead* of solitons. Creates decomposition based on eigenvalue factorization of τ:

$$f_1 = 2\partial_x^2 \log\left(e^{2\eta_1} + e^{2\eta_2} + 2\epsilon^2 e^{2(\eta_1+\eta_2)} - \sqrt{\gamma}\right)$$

$$f_2 = 2\partial_x^2 \log\left(e^{2\eta_1} + e^{2\eta_2} + 2\epsilon^2 e^{2(\eta_1+\eta_2)} + \sqrt{\gamma}\right)$$

$$\gamma = e^{4\eta_1} + e^{4\eta_2} - \frac{2(k_1^2 - 6k_1k_2 + k_2^2)}{(k_1+k_2)^2} e^{2(\eta_1+\eta_2)}.$$





Key:

— $= f_1(x,t)$

— $= f_2(x,t)$

Properties: order preserving, $\pm$
formula not too nice or natural
not spacetime symmetric!

Vain Remarks

Note that only our decomposition has all three of these "soliton like" properties:

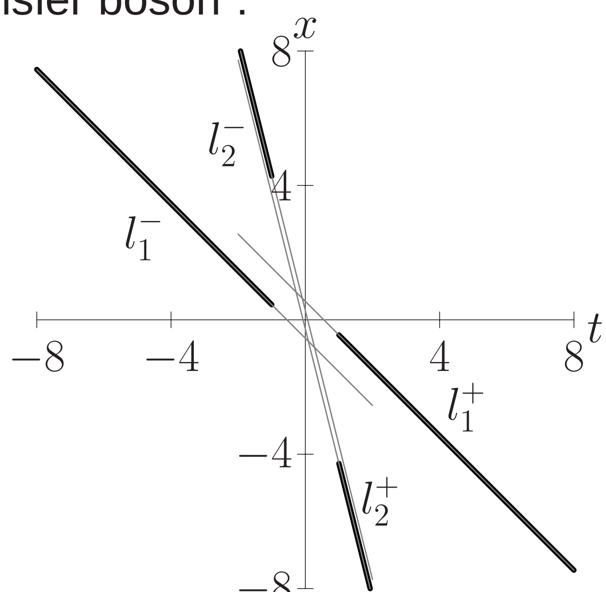
- All of its elements are all non-negative, taking only strictly positive values when the parameters and variables are real.
- The set itself is closed under the involution $x \longrightarrow -x$ and $t \longrightarrow -t$, which is to say that if one is watching a KdV soliton interaction or the same thing shown in a mirror and run backwards in time.
- All of its elements take the form of quotients of finite linear combinations of the form $\exp(ax + bt)$.

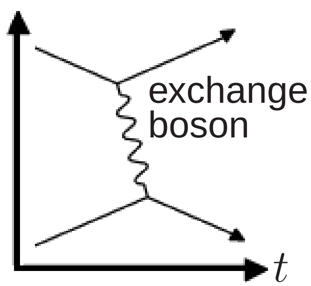
Next: Decompositions into Three or More Parts

$$u_2(x,t) = f_1(x,t) + f_2(x,t) + f_3(x,t) + \cdots$$

Why consider $n > 2$?

Argument #1: The timing of asymptote intersections suggests "transfer boson":





Argument #2: Lax's original paper discusses the number of local maxima in 2-soliton solution as function of the speeds k_1 and k_2. All have 2 local maxima for almost all times but:

- If k_1/k_2 is large: there is a moment with just one maximum.
- If k_1/k_2 is small: two local maxima at *all* times.
- In between: there is a moment when there are *three* maxima.

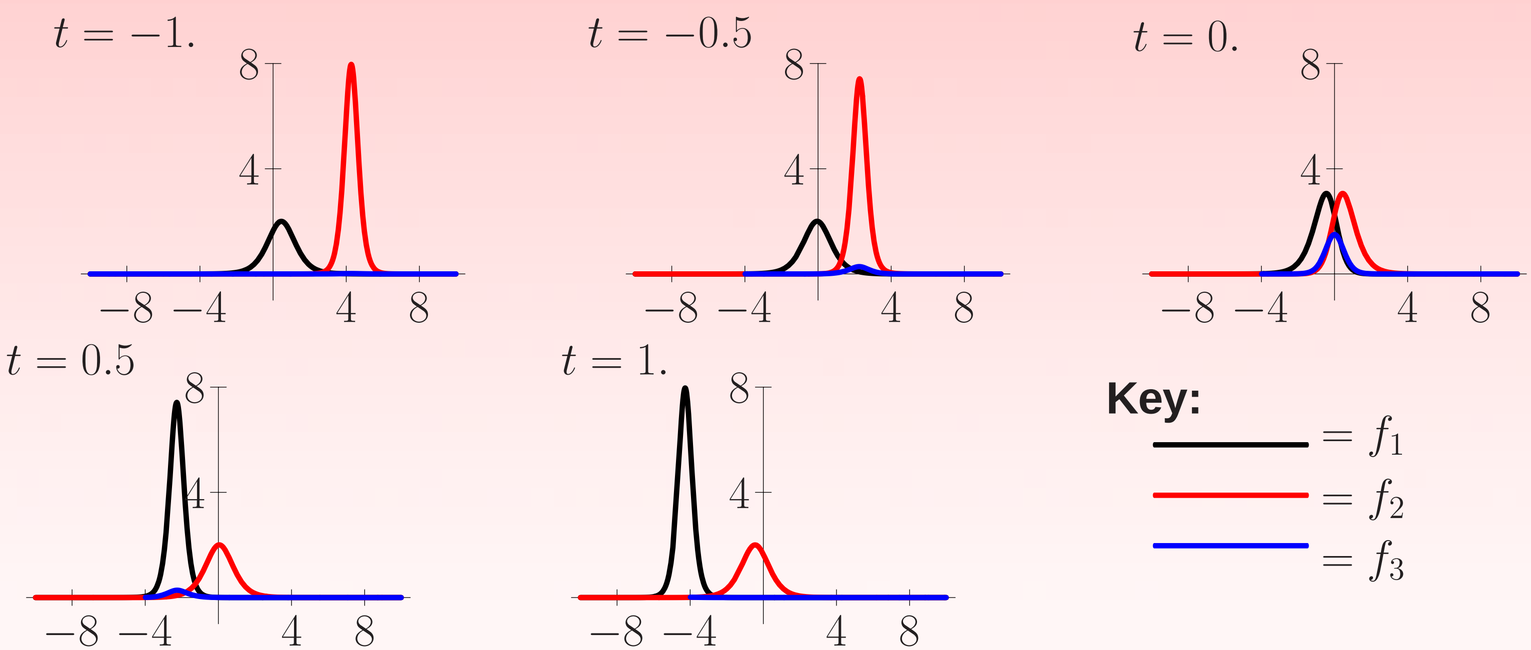
Bryan and Stuart's 3-part decomposition

Their decomposition also starts with eigenvalues of same matrix as Nguyen, so γ is the same:

$$f_i = 2\frac{(\mu_i')^2}{\mu_i(1+\mu_i)^2} \qquad i = 1, 2 \qquad f_3 = \sum_{i=1}^{2}(2\partial_x^2 \ln(\mu_i))\frac{\mu_i}{1+\mu_i}$$

where

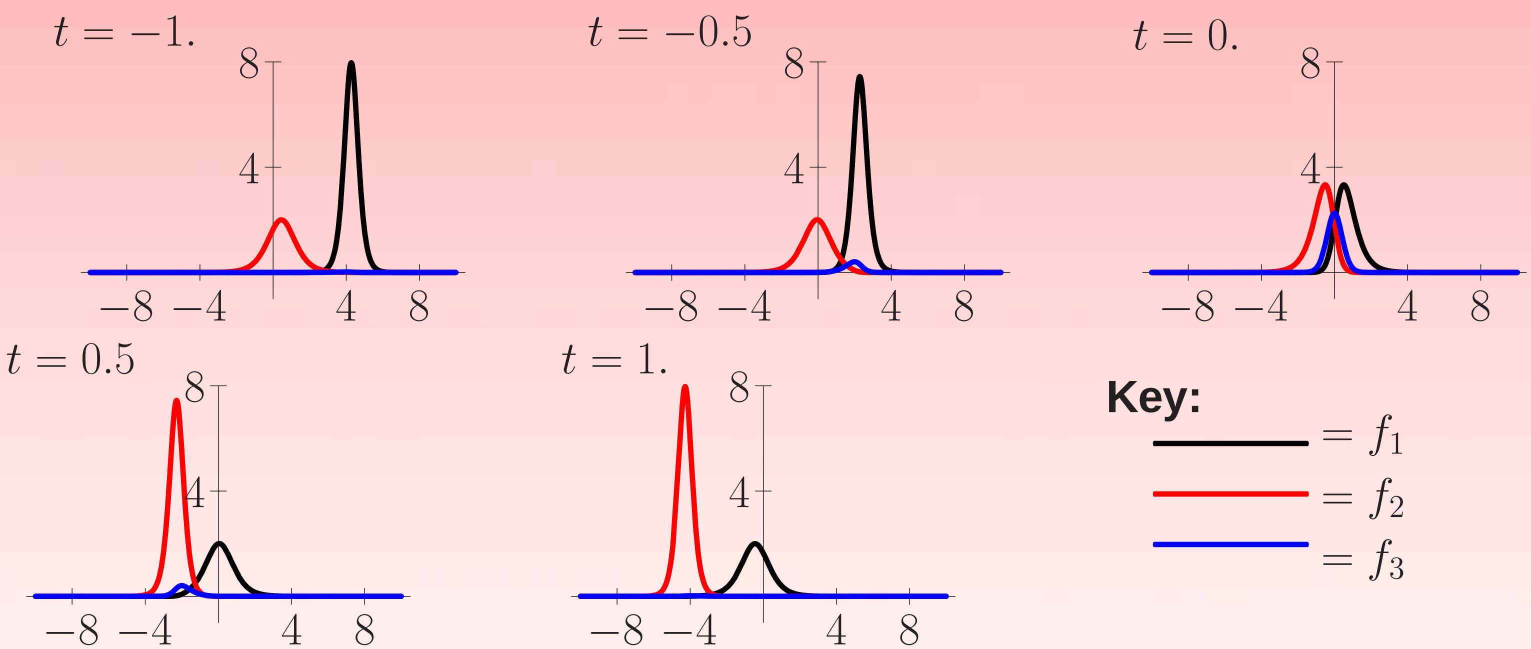
$$\mu_i = \frac{(k_1+k_2)e^{-2\eta_1-2\eta_2}}{2(k_2-k_1)^2}\left(e^{2\eta_1}+e^{2\eta_2}+(-1)^i\sqrt{\gamma}\right)$$



Our decomposition with “exchange soliton”

$$f_1(x,t) = \frac{8\epsilon^2(k_2^2 e^{2\eta_1} + k_1^2 e^{2\eta_2})}{\tau^2} \qquad f_2(x,t) = \frac{8(k_2^2 e^{-2\eta_1} + k_1^2 e^{-2\eta_2})}{\tau^2}$$

$$f_3(x,t) = \frac{16(k_2 - k_1)^2}{\tau^2}$$



Here, f_3 vanishes for $|t| \to \infty$ and has a unique local max $\forall t$ located at $x = -\frac{1}{k_2}(k_2^3 t + \xi_2 + \log\sqrt{\epsilon})$.

Conclusions and Outlook

- Nguyen even has a decomposition of u_2 with *four* parts!
- Question of how to identify the solitons before and after the interactions is not well posed mathematical problem: one should not be expecting a definitive answer.
- Other ways: Several authors have attempted to provide motivation for the order preserving interpretation by reference to moving “point particles” associated to singularities of solutions of the KdV equation.
- Making new out of old: If $\{f_i\}$ and $\{g_i\}$ are decompositions of u_2 then so is $\{F(x,t)f_i + (1 - F(x,t))g_i\}$ for an *arbitrary* function F. (This dramatically demonstrates the extent to which the decompositions fail to be unique.)
- Future goals: Decomposition of n-soliton; Decomposition of KP soliton, find explicit connection between “exchange soliton” and process of “bosonization”.

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Preprint: https://aps.arxiv.org/abs/nlin.PS/0602036

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