Bispectrality and Duality

Alex Kasman, College of Charleston
Joint Mathematics Meeting - Boston 2012

What is *bispectrality*?

Spectral Parameter

We often consider *families of eigenfunctions* for Lax operators for which the eigenvalue is depends upon an extra parameter. For example

$$L = \partial_x^2 - \frac{2}{x^2} \qquad \psi(x,z) = \left(1 - \frac{1}{xz}\right) e^{xz} \qquad L\psi = z^2\psi. \qquad (*)$$

Bispectrality

In some cases, the same eigenfunction satisfies a pair of eigenvalue equations with the roles of spatial and spectral parameters switched.

Definition: $(L_x, \Lambda_z, \psi(x,z))$ is a bispectral triple if

$$L\psi(x,z) = p(z)\psi(x,z) \qquad \text{and} \qquad \Lambda\psi(x,z) = \pi(x)\psi(x,z).$$

Example: $L = \partial^2$, $\Lambda = z\partial_z$, $\psi = z^x$: $L\psi = (\ln z)^2\psi \qquad \Lambda\psi = x\psi$

Example: Example $(*)$ above is trivially bispectral since $\psi(x,z) = \psi(z,x)$.

Bispectrality: Schrödinger Case

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Theorem 0.1. *The potentials V for which* (0.1), (0.2) *hold (for non-zero ϕ and A of positive order) are $V(x)=\alpha x+\beta$, $\alpha,\beta\in\mathbb{C}$, $\alpha\neq 0$ (Airy) or $V(x)=\dfrac{c}{(x-a)^2}+b$, $a,b,c\in\mathbb{C}$ (Bessel) or, modulo a translation in x and adding a constant to V, those which can be obtained from $V=0$ or $V=-\dfrac{1}{4}\dfrac{1}{x^2}$ by finitely many rational Darboux transformations*

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- More interesting: can be summarized by saying the potential must be a rational KdV solution! (Dynamics or coincidence?)

Bispectrality: Rank 1 Case

- G. Wilson (1993) completely characterized the set of all bispectral triples (L,Λ,Ψ(x,z)) **where *L* commutes with other odos of relatively prime order**. (Answer: iff spectral curve is rational with only cuspidal singularities.)

- *Wilson made use of the known correspondence between such operators with solutions to the KP equation.*

- Turns out that Λ commutes with relatively prime order too, so we have $\Psi(x,z) \Rightarrow \Psi(z,x)$ (bispectral involution)

What is Classical Duality?

What is a particle system?

Consider the positions x_i and momenta y_i of n particles as functions of time t.

The Hamiltonian function $H(x_1, \ldots, x_n, y_1, \ldots, y_n)$ determines their dynamics according to

$$\frac{\partial x_i}{\partial t} = \frac{\partial H}{\partial y_i} \qquad \frac{\partial y_i}{\partial t} = -\frac{\partial H}{\partial x_i}.$$

What is integrability?

Only for rare choices of H are there explicit $x_i(t)$ and $y_i(t)$. In those cases, there is a function ("symplectic map") $F : (x_i, y_i) \to (X_i, Y_i)$ such that $\dot{X}_i = 0$ and $\dot{Y}_i$ are constant.

In essence, this F^{-1} takes simple "linear" dynamics and twists it into a complicated looking rule.

Duality:

Integrable particle systems have a natural "duality": pair the system with linearizing map F with the one that has F^{-1}!

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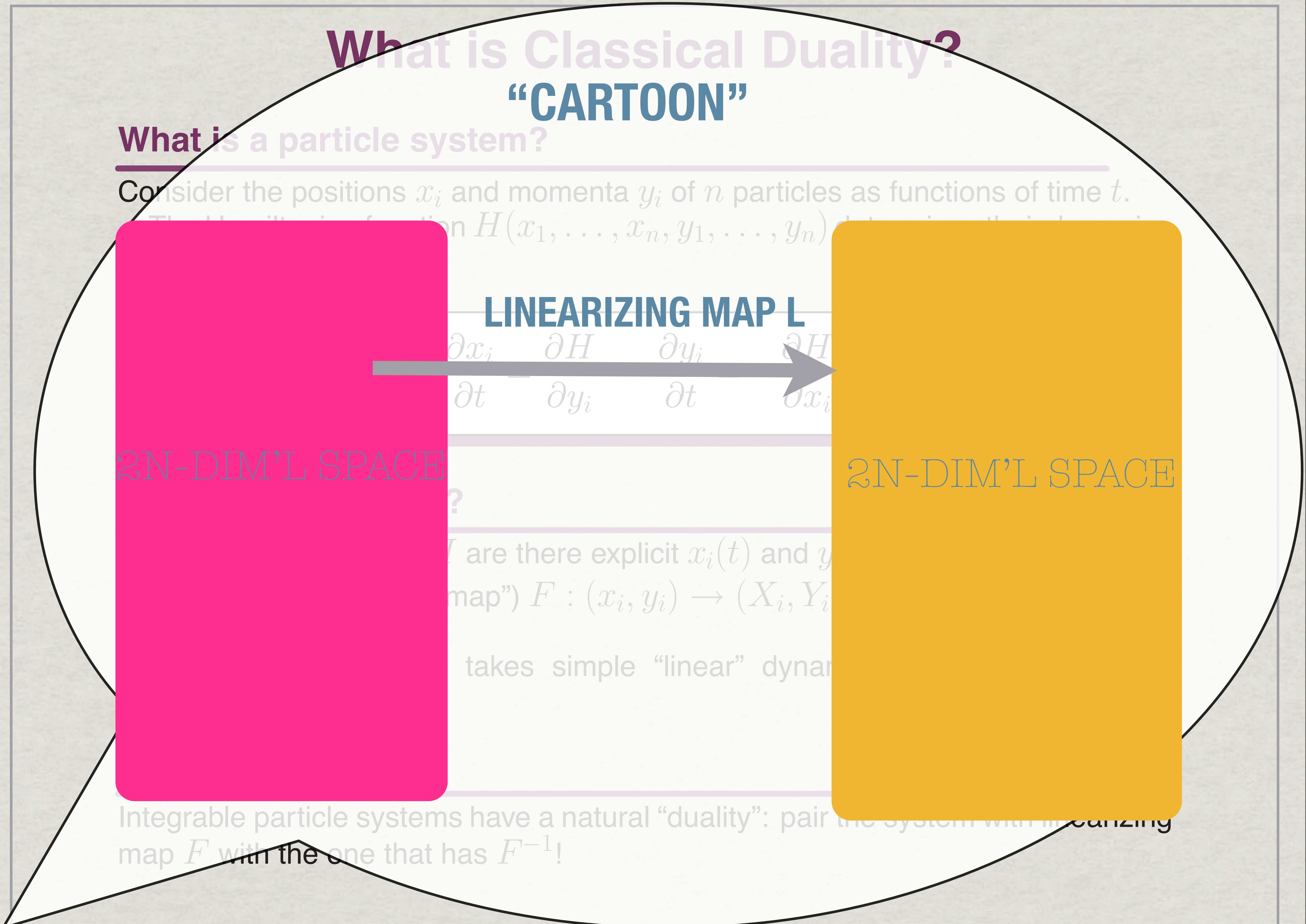

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“CARTOON OF DUALITY”

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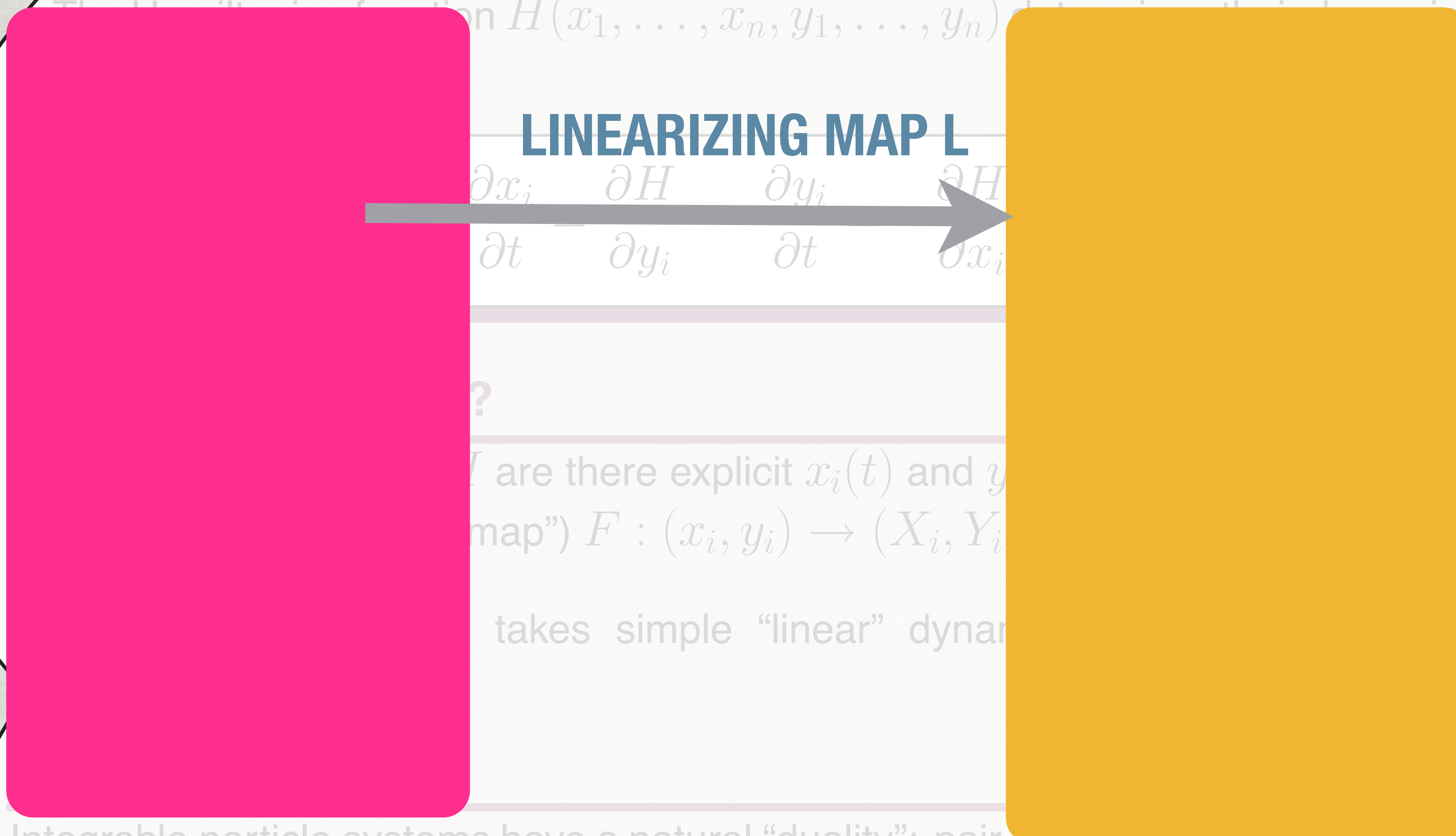

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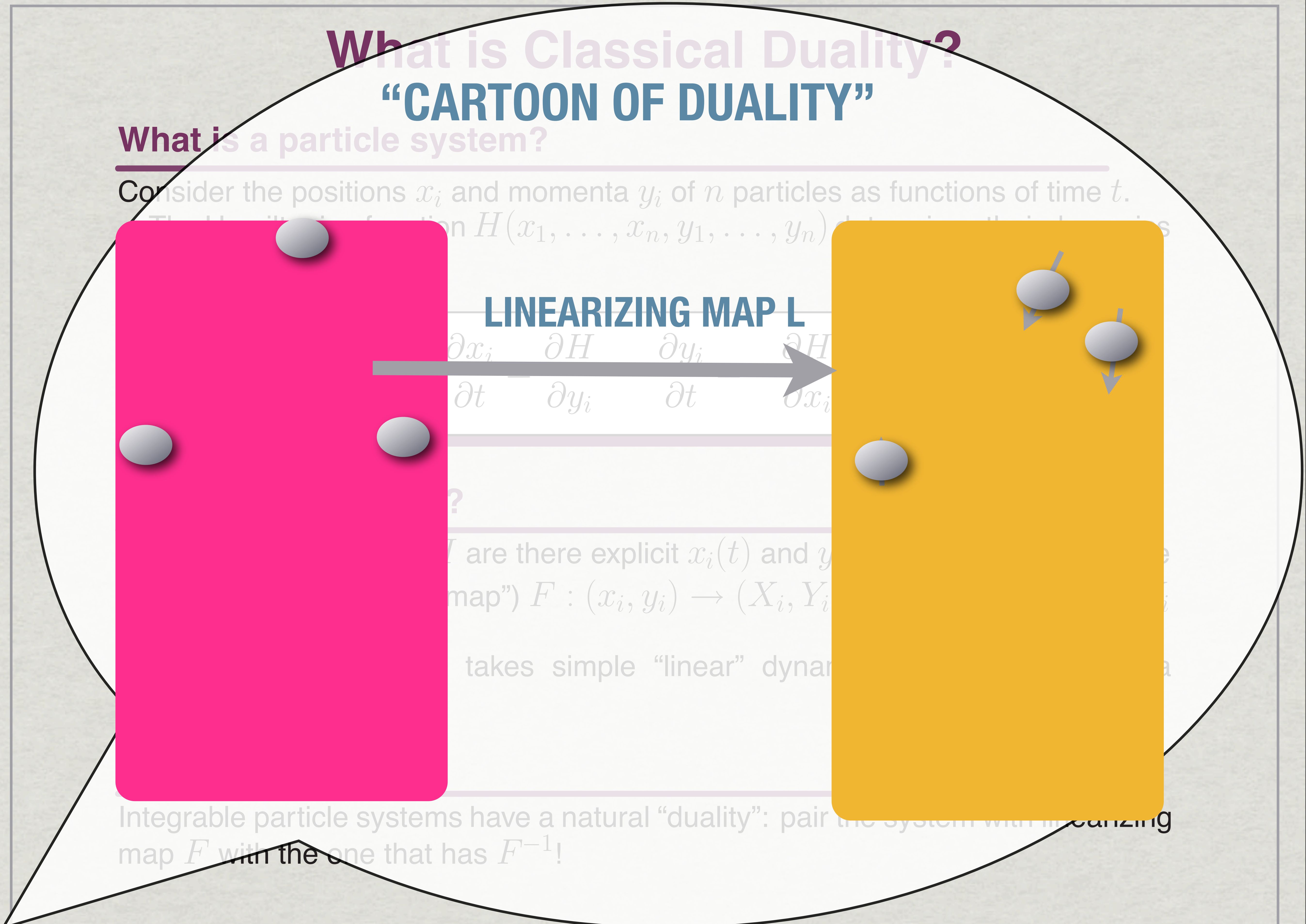

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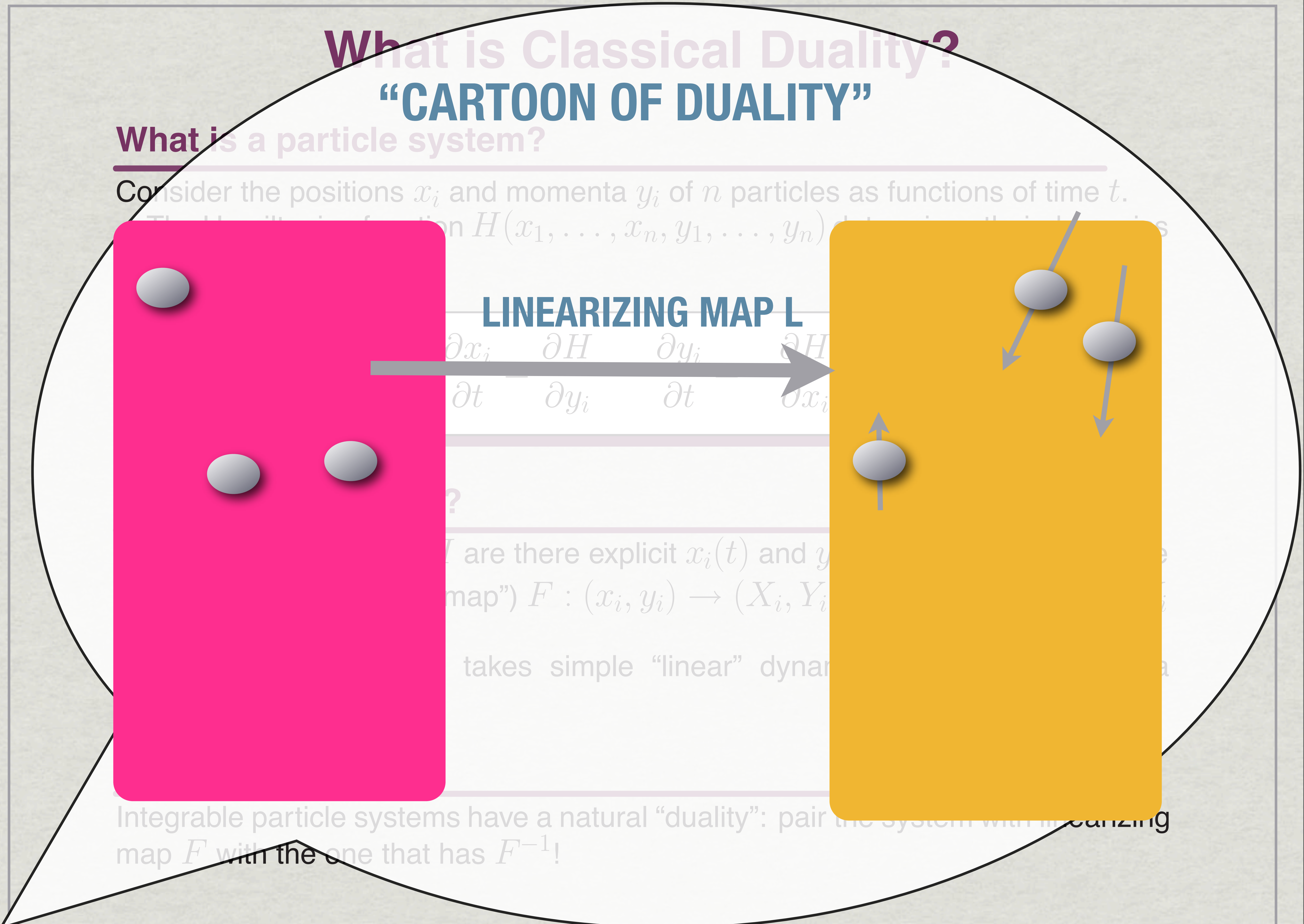

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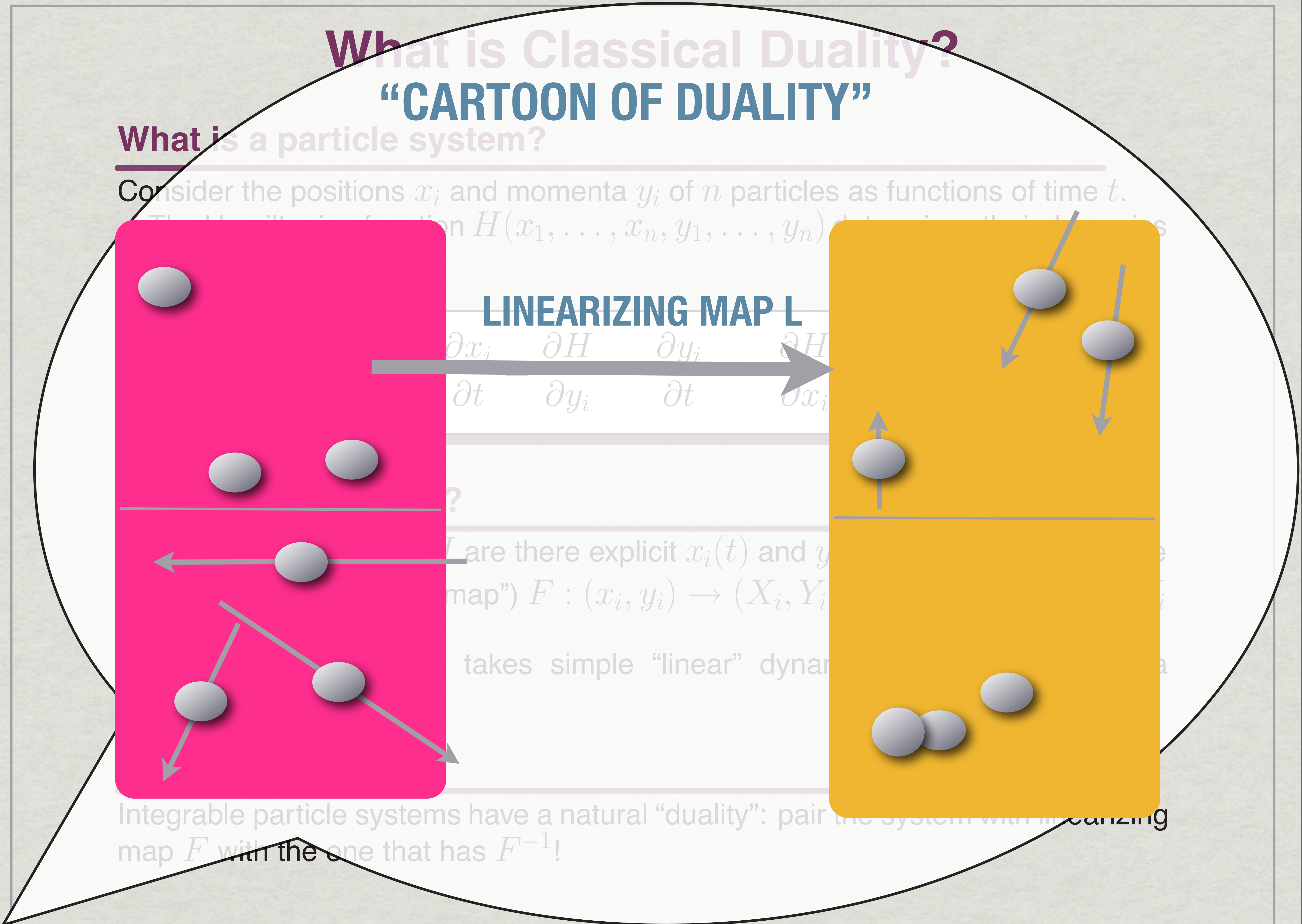

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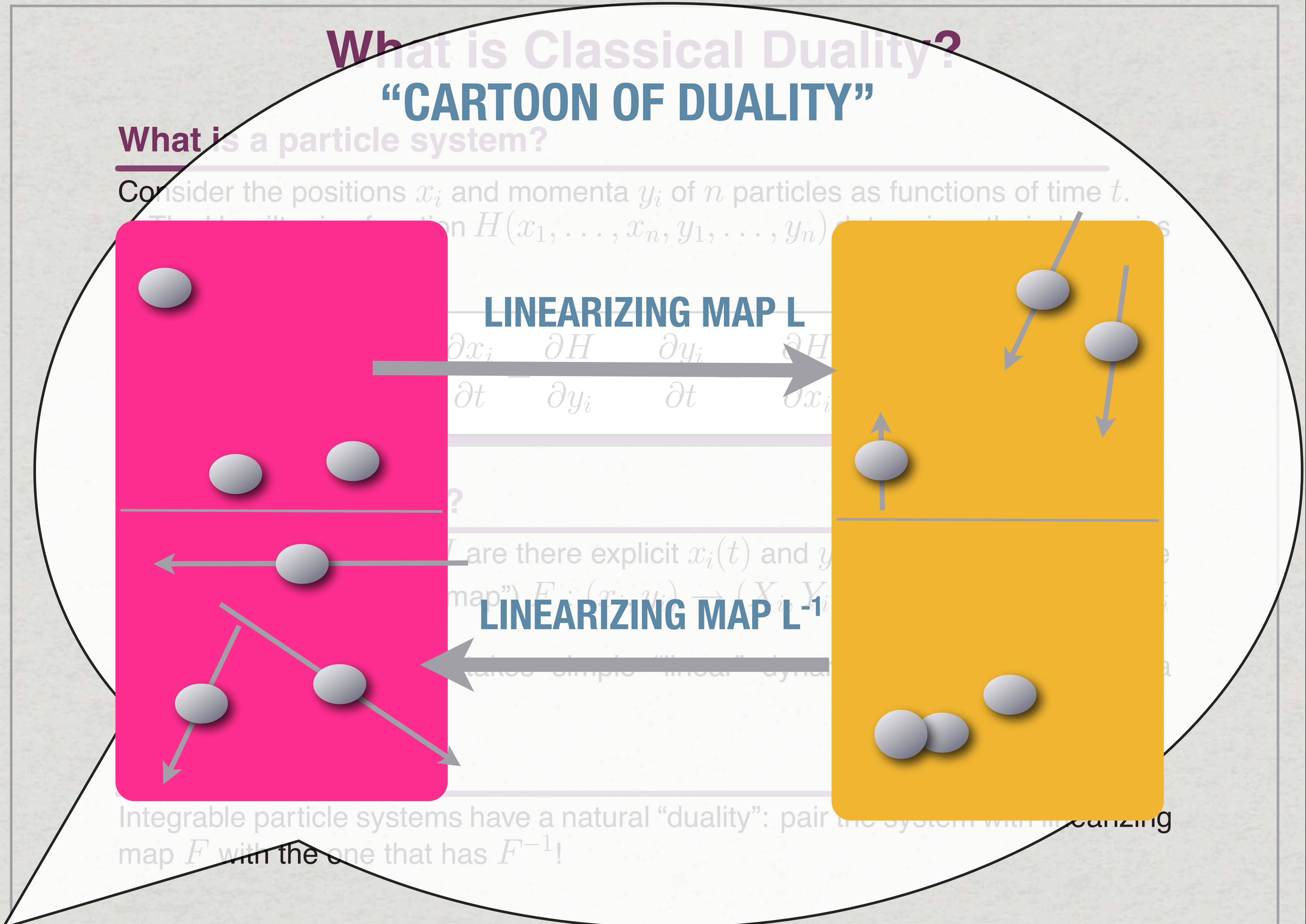

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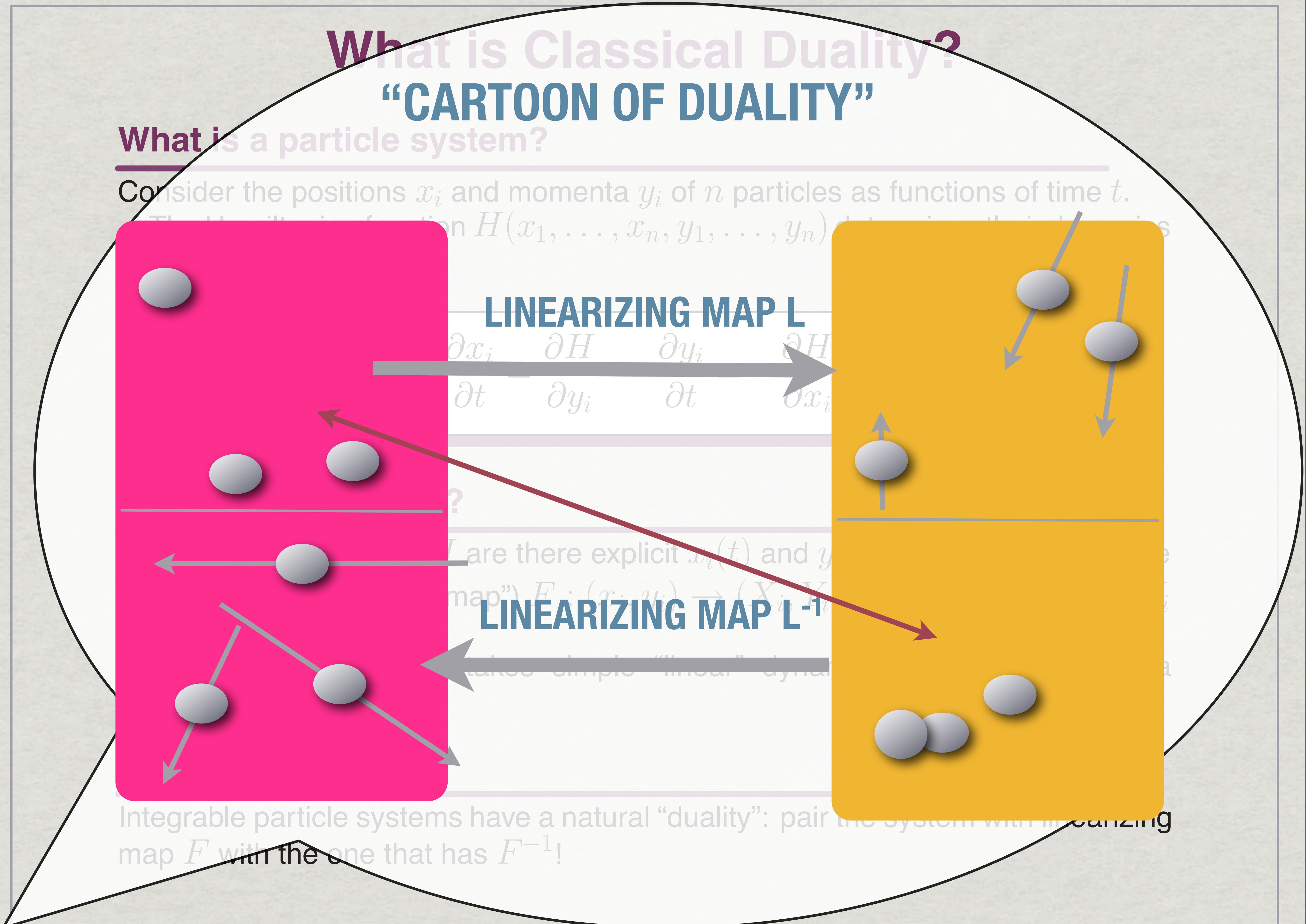

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Example: Calogero System

In the early 1970's, F. Calogero showed that the Hamiltonians

$$H_k = \mathrm{tr} M^k \qquad M_{ij} = y_i \delta_{ij} + \frac{1-\delta_{ij}}{x_i - x_j}$$

are integrable. This system is known to govern pole dynamics for soliton equations. Its quantum analogue shows extreme exclusion statistics.

Interestingly, J. Moser showed that their linearizing map is an *involution.* <u>This system is *self-dual!*</u>

(Non-self dual example: Ruijsenaars-Schneider is dual to *hyperbolic* Calogero-Moser.)

Quantum Duality=Bispectrality

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It is remarkable that the BA function turns out to be symmetric with respect to k and x. For Coxeter configurations this property has been established in [5].

Theorem 2.3. *Baker-Akhiezer function* $\psi(k,x)$ *is symmetric with respect to* x *and* k: $\psi(k,x) = \psi(x,k)$.

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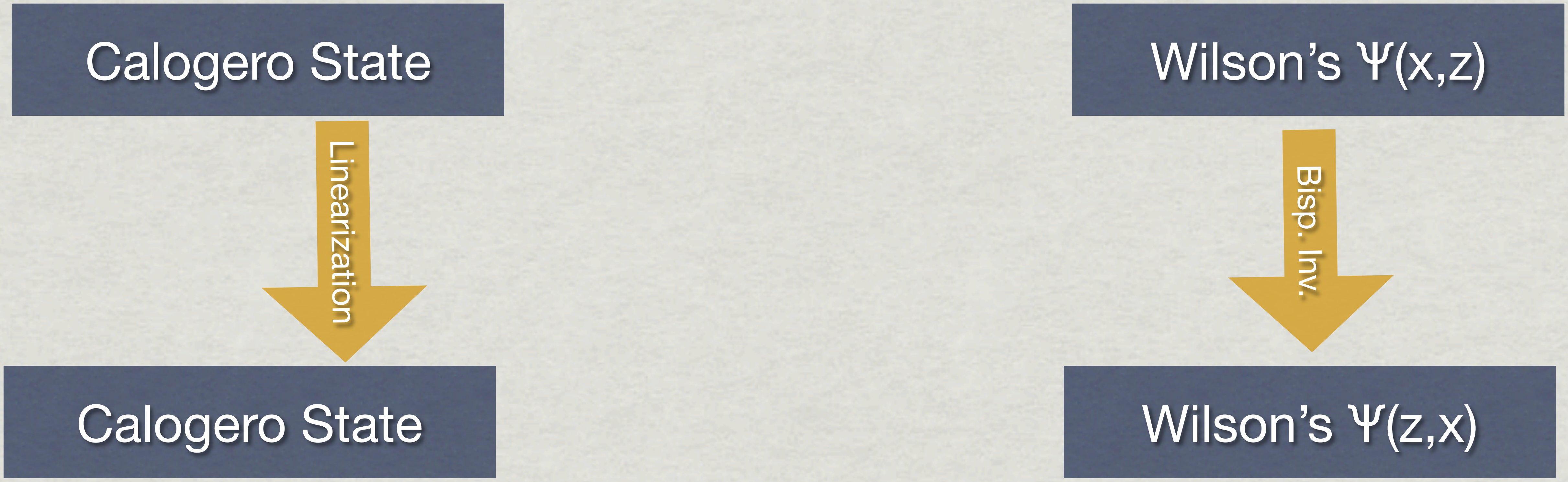

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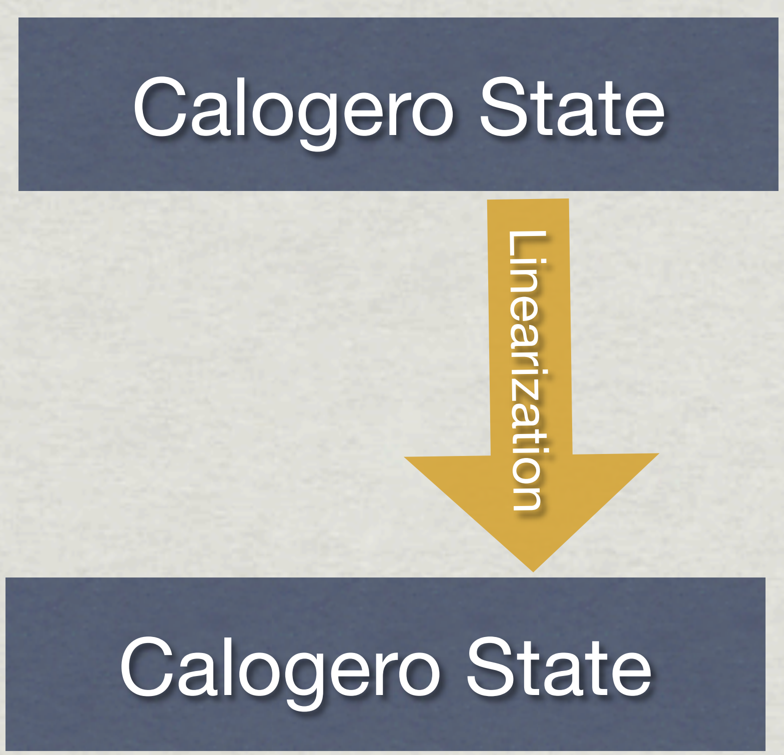

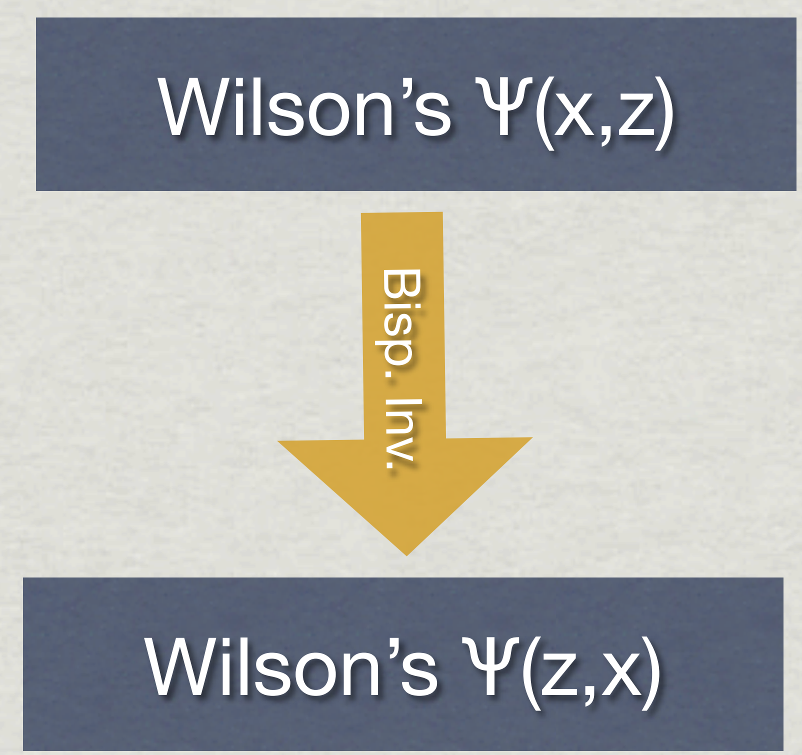

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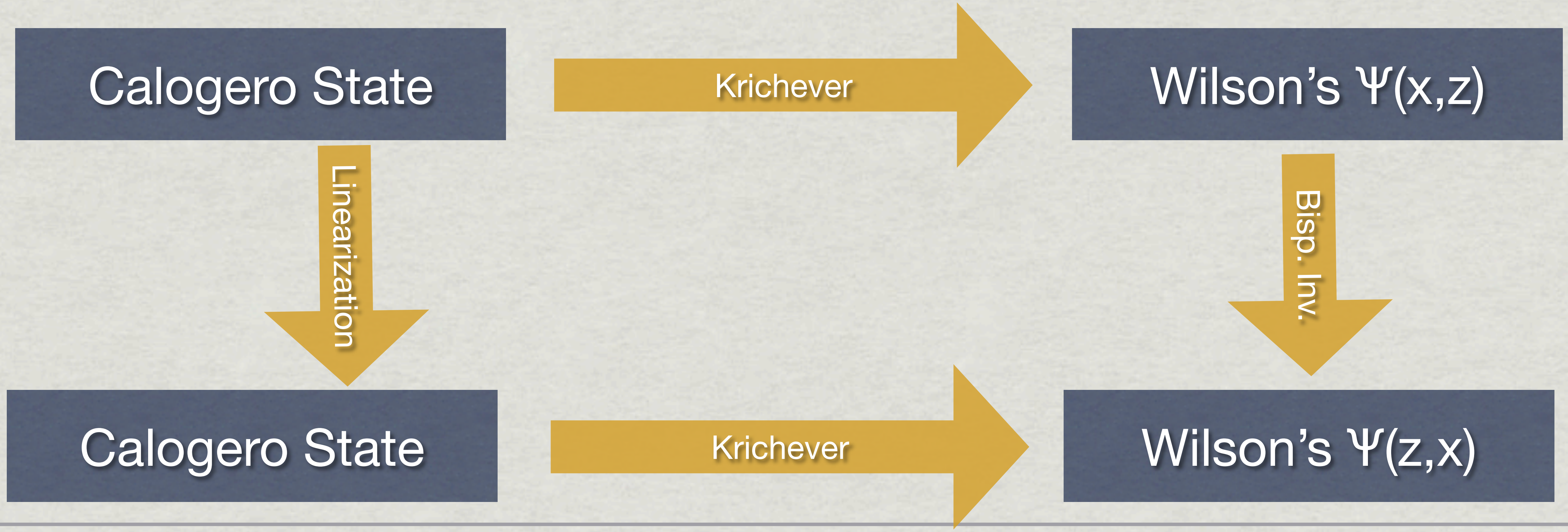
- I (unjustifiably) felt clever when I showed that this diagram commutes:



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Commun. Math. Phys. 172, 427–448 (1995)

Communications in Mathematical Physics

Bispectral KP Solutions and Linearization of Calogero–Moser Particle Systems

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Abstract: Rational and soliton solutions of the KP hierarchy in the subgrassmannian Gr_1 are studied within the context of finite dimensional dual grassmannians. In the rational case, properties of the tau function, τ, which are equivalent to bispectrality of the associated wave function, ψ, are identified. In particular, it is shown that there exists a bound on the degree of all time variables in τ if and only if ψ is a rank one bispectral wave function. The action of the bispectral involution, β, in the generic rational case is determined explicitly in terms of dual grassmannian parameters. Using the correspondence between rational solutions and particle systems, it is demonstrated that β is a linearizing map of the Calogero-Moser particle system and is essentially the map σ introduced by Airault, McKean and Moser in 1977 [2].

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- Contained an important technique: **rank one operator identities.**

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Bispectrality=Duality Program Overview

Quantum

Classical

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In joint paper with Emil Horozov (1998) *used* bispectrality/duality correspondence to produce new dual quantum Hamiltonian pairs from any given example.

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Calogero Self-duality = bispectrality of scalar ODOs which commute with others of relatively prime order (Kasman ’95 / Wilson ‘98)

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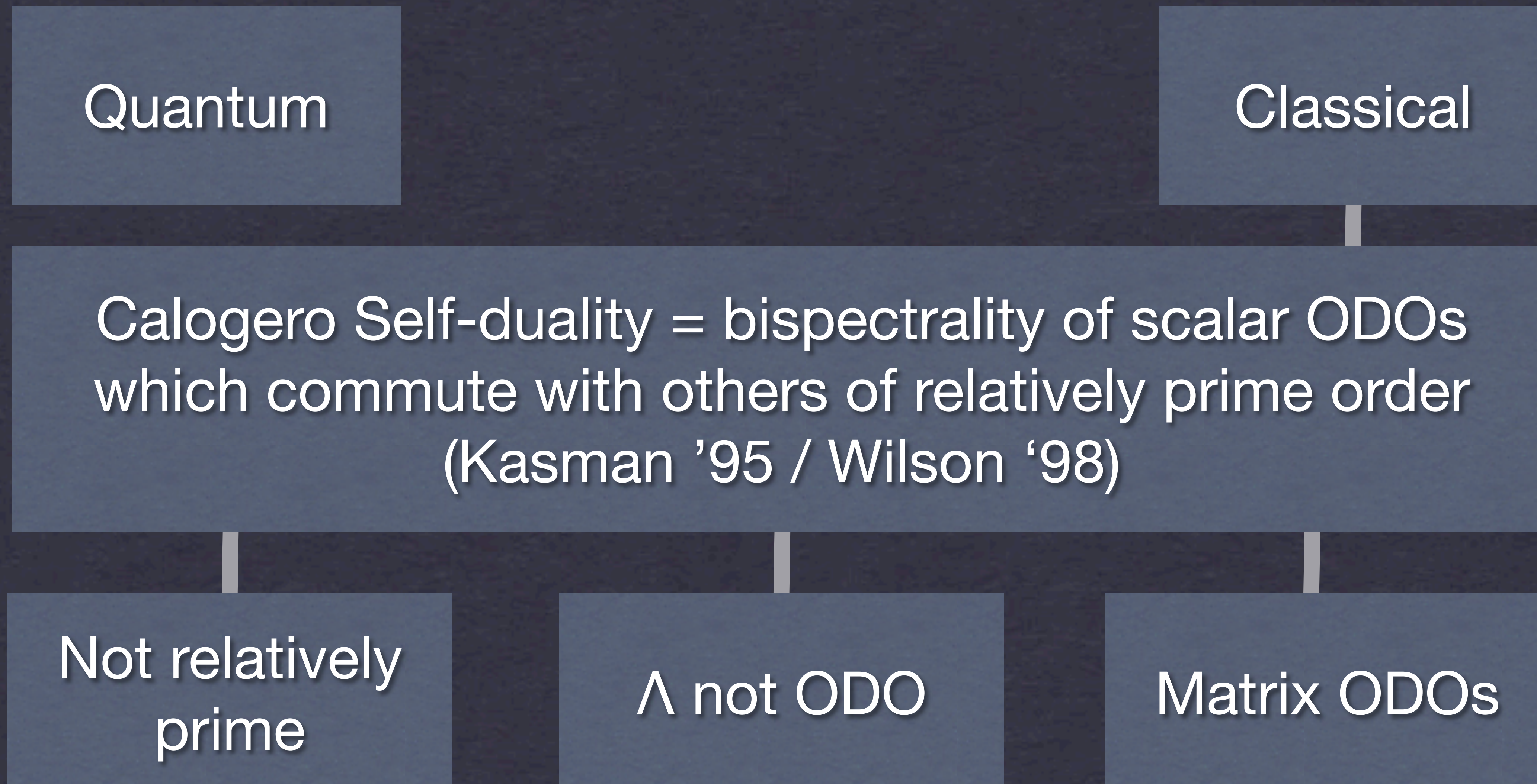

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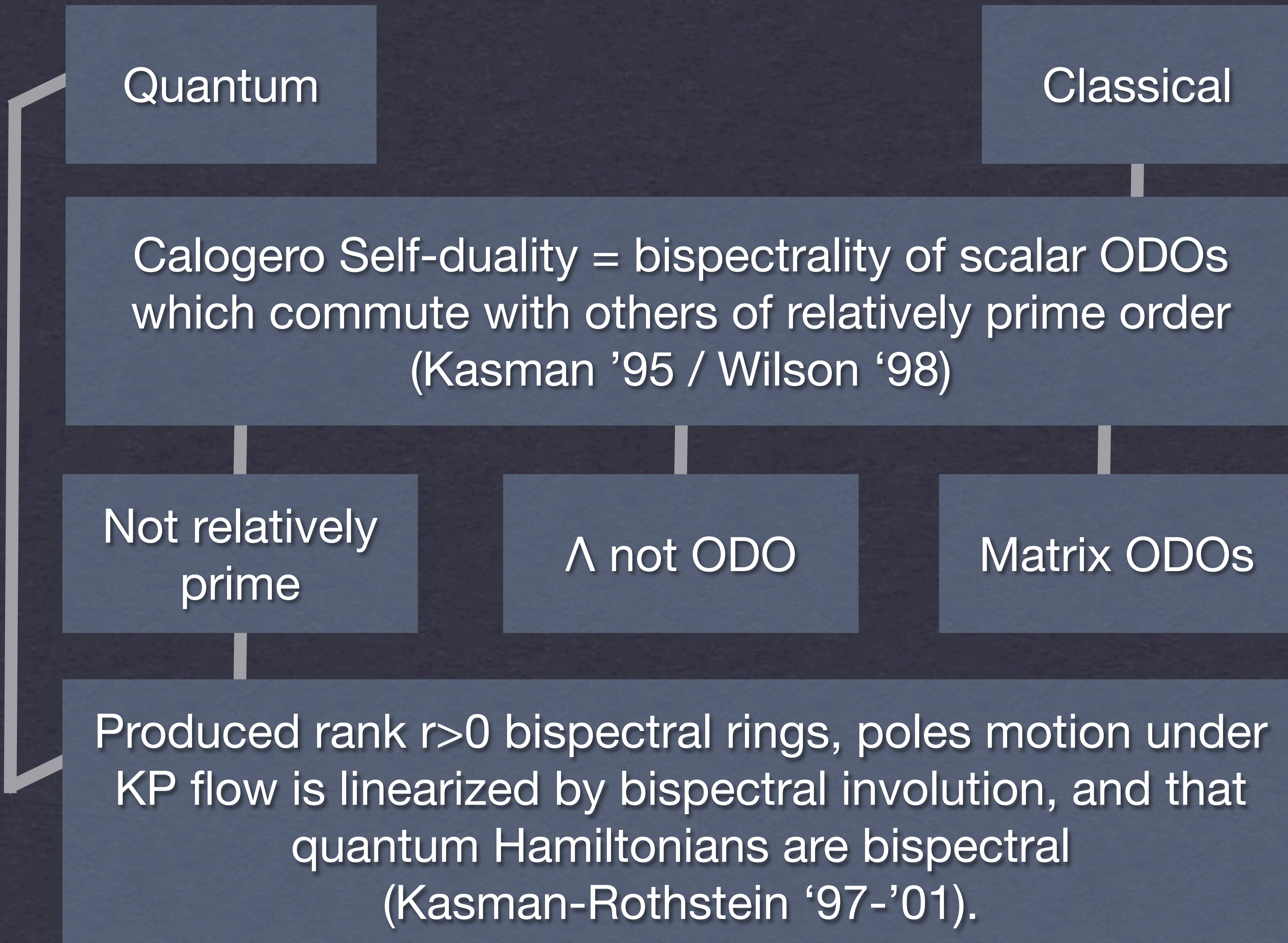

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Calogero Self-duality = bispectrality of scalar ODOs which commute with others of relatively prime order (Kasman ’95 / Wilson ‘98)

Not relatively prime

Λ not ODO

Matrix ODOs

I was planning ahead: Bispectrality for Solitons (translation operators) in 1998, Rank One Formulas for Solitons (w/Gekhtman) in 2001...working towards duality

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In 2007, Haine used these matrices and operators to show that *indeed* the bispectral involution for KP solitons is the action-angle map for Ruijsenaars/Hyperbolic Calogero. (Non-self dual case!)

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Old paper of Zubelli suggests that bispectrality and matrices don't mix. So, nobody looked at it much after.

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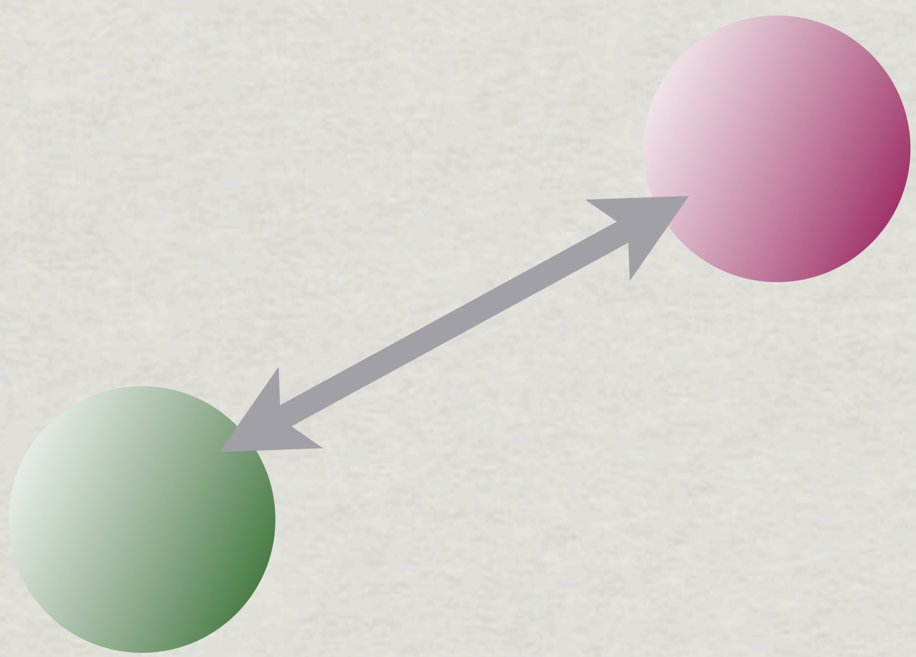
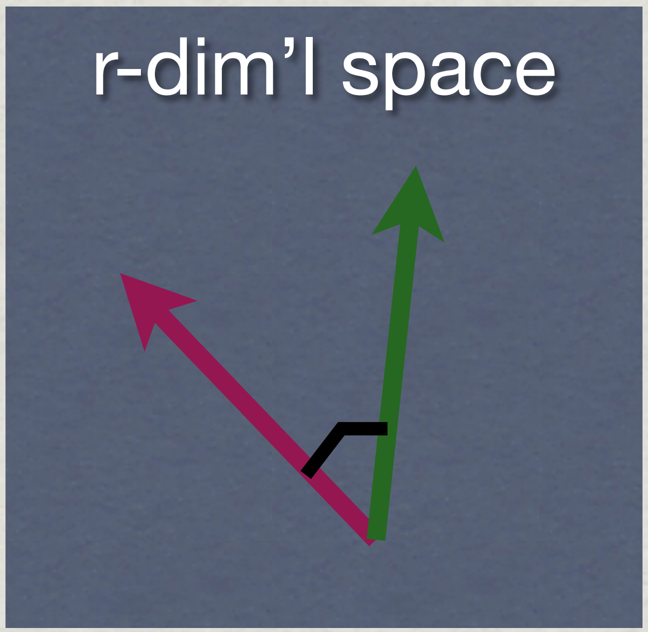
Some details from *Mathematical Physics, Analysis and Geometry* 12 (2009) 181-200 [Bergvelt-Gekhtman-K].

A NEW “SPIN” ON PARTICLE DYNAMICS

In addition to position and momentum, the interaction between each pair of particles i and j now depends on the number $s_{ij} = \alpha_i \cdot \beta_j$ where α_i and β_j are r-vectors. (We assume $s_{ii} = s_{jj}$.)

Note that $R = (s_{ij})$ is a matrix of rank r. (In hindsight, the “rank one conditions” from many of my previous papers was a “spinless” assumption.)





Spin Calogero

Let

$$X_{ij} = x_i \delta_{ij} \qquad Z_{ij} = y_i \delta_{ij} + (1 - \delta_{ij}) \frac{s_{ij}}{x_i - x_j}.$$

The eigenvalues dynamics of $X + ktZ^{k-1}$ are governed by $H_k = \text{tr} Z^k$.
Note that the rank r condition "$[X, Z] - I = R$" holds.
More generally,

$$\text{sCM}_r^n = \{(X, Z, A, B) \mid X, Z \in M_{n\times n},\ A, B^\top \in M_{r\times n},\ [X, Z] - I = BA \neq 0\}$$

is the state space of the spin Calogero system (including particle "collisions").
The linearizing map is the involution

$$(X, Z, A, B) \mapsto (Z^\top, X^\top, B^\top, A^\top).$$

All of that is old news. What we need to do now is show that there is a natural way to associate a bispectral matrix KP solution to each point of sCM_r^n (generalizing the known $r = 1$ case), and that the linearizing map corresponds to the bispectral involution.

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Bispectrality for Matrices...with a "twist"

- Not much has been done with bispectral matrix odos since Zubelli (1989). Seems unlikely that there are so many "unnoticed" bispectral matrix operators.
- His "bispectral problem" was in the form

$$L\psi = \sum_{i=0} M_i(x) \frac{\partial^i}{\partial x^i} \psi(x,z) = p(z)\psi(x,z)$$

$$\Lambda\psi = \sum_{i=0} \hat{M}_i(z) \frac{\partial^i}{\partial z^i} \psi(x,z) = \pi(x)\psi(x,z).$$

- This turns out not to be terribly rich. As we'll see, the generalization of Wilson's result requires us to look at

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Result 1: Analogue of Krichever Map

Definition: To (X, Z, A, B) associate the $r \times r$ matrix odo ($x = t_1$)

$$W = \det(\partial I - Z)I + A(X - \sum it_i Z^{i-1})^{-1} adj(Z - \partial I)B.$$

Theorem: $\mathcal{L} = W \circ \partial \circ W^{-1}$ is a solution to the KP hierarchy.

Key Steps of Proof:

- Using matrix analysis and the rank r condition, we show that the kernel of W can be written nicely in terms of the residues of $e^{\sum t_i z^i} / \det(zI - Z)$ at the eigenvalues of Z.

- We then differentiate $W\phi = 0$ wrt t_i and derive the "Lax equation"

$$\dot{\mathcal{L}} = [\mathcal{L}, (\mathcal{L}^i)_+]$$

  from it using differential algebra.

Remark: Wilson's $r = 1$ proof was similar, but was only handled Z with distinct eigenvalues. This proof “fills the hole”!

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Note that this is the matrix whose eigenvalue dynamics are governed by Spin Calogero...and hence so are the pole dynamics of the KP solution.

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Result 2: Bispectrality

Definition: For each choice of (X, Z, A, B) and $\mathcal{L} = W \circ \partial \circ W^{-1}$ as before we define $L = p(\mathcal{L})$ where $p(z) = z^k \det(zI - Z)^2$ ($k \geq 0$).

Theorem (Eigenfunction) : L is an *ordinary* differential operator satisfying

$$L\psi = p(z)\psi \qquad \text{for} \qquad \psi(x,z) = e^{xz}\left(I + A(xI - X)^{-1}(zI - Z)^{-1}B\right).$$

Since this works for $k = 0$ and $k = 1$, we have commuting operators of relatively prime order.

Definition: Of course, we could do the same for $(Z^\top, X^\top, B^\top, A^\top)$ and get a different differential operator $L^\flat$ satisfying

$$L^\flat \psi^\flat(x,z) = \pi(z)\psi^\flat(x,z).$$

Theorem (Bispectral Involution): We use the simple fact that

$$\psi(x,z) = \left(\psi^\flat(z,x)\right)^\top$$

to conclude that $\Lambda = \left(L^\flat|_{x\to z}\right)^\top$ satisfies

$$\Lambda_R \psi(x,z) = \pi(x)\psi(x,z).$$

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- Can we *prove* that dual Hamiltonians quantize to bispectral operators? (Fock-Gorsky-Nekrasov-Rubtsov)
- Can we prove that bispectral involutions and action-angle maps for dual integrable systems are always the same?
- Why does duality look like bispectrality for *both* quantum and classical systems? (Note: At the quantum level, the Hamiltonians are bispectral and classically it is individual states that are!)